

Objective Bayesian Analysis for the Lomax Distribution

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Abstract

In this paper we propose to make Bayesian inferences for the parameters of the Lomax distribution using non-informative priors, namely the Jeffreys prior and the reference prior. We assess Bayesian estimation through a Monte Carlo study with 500 simulated data sets. To evaluate the possible impact of prior specification on estimation, two criteria were considered: the bias and square root of the mean square error. The developed procedures are illustrated on a real data set.

Keywords: Bayesian inference, Jeffreys prior, Lomax distribution, reference prior.

1 Introduction

The Lomax distribution [17], also known as the Pareto Type II distribution (or simply Pareto II), is a heavy-tail probability distribution often used in business, economics and actuarial modeling. It is essentially a Pareto distribution that has been shifted so that its support begins at zero [22]. The Lomax distribution has been applied in a variety of contexts ranging from modeling the survival times of patients after a heart transplant [1] to the sizes of computer files on servers [14]. Some authors, such as [9], suggest the use of this distribution as an alternative to the exponential distribution when data are heavy-tailed.

The main objective of this paper is to make Bayesian inferences for the parameters of the Lomax distribution using non-informative priors, namely the Jeffreys prior [16] and the reference prior [7]. Then, we perform a simulation study to compare the efficiency of the Bayesian approach for estimating the model parameters under these two priors, and check for the possible impact of prior specification. We also show how to represent the Lomax distribution in a hierarchical form by augmenting the model with a latent variable which makes the Bayesian computations easier to implement. This would also allow

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the user to implement inferences using all-purpose Bayesian statistical packages like WinBUGS [18] or JAGS [19].

The remainder of this paper is organized as follows. In Section 2, we present the Lomax distribution and list some of its properties. In Section 3, we formulate the Bayesian model using non-informative priors. In Section 4, a simulation study is presented. In Section 5, the methodology is illustrated on a real data set. Some final comments are given in Section 6.

2 Model definition

Here, we use the definition that appears, for example, in [15].

Definition 2.1. *A continuous random variable X has a Lomax distribution with parameters α and β if its probability density function is given by*

$$f(x|\beta, \alpha) = \frac{\alpha}{\beta} \left(1 + \frac{x}{\beta}\right)^{-(\alpha+1)}, \quad x \geq 0,$$

where $\alpha > 0$ and $\beta > 0$ are the shape and scale parameters, respectively.

We refer to this distribution as *Lomax* (β, α) . The median is $\beta(2^{1/\alpha} - 1)$ and the mode is zero. The hazard function is given by

$$h(x|\beta, \alpha) = \frac{\alpha}{\beta} \left(1 + \frac{x}{\beta}\right)^{-1}, \quad x \geq 0,$$

which is a decreasing function of x , thus making this a suitable model for components that age with time. The survival function is given by

$$S(x|\beta, \alpha) = \left(1 + \frac{x}{\beta}\right)^{-\alpha}, \quad x \geq 0.$$

We note also that the Lomax distribution can be expressed in the following hierarchical form

$$\begin{aligned} X|\beta, \lambda &\sim \text{Exponential}\left(\frac{\lambda}{\beta}\right), \\ \lambda|\alpha &\sim \text{Gamma}(\alpha, 1). \end{aligned}$$

This follows from writing the joint density of X and λ as

$$f(x|\beta, \lambda)f(\lambda|\alpha) = \frac{1}{\beta\Gamma(\alpha)}\lambda^\alpha \exp\left\{-\lambda\left(1 + \frac{x}{\beta}\right)\right\}.$$

So, the marginal density of X is given by

$$\begin{aligned} f(x) &= \frac{1}{\beta\Gamma(\alpha)} \int_0^\infty \lambda^\alpha \exp\left\{-\lambda\left(1 + \frac{x}{\beta}\right)\right\} d\lambda \\ &= \frac{1}{\beta\Gamma(\alpha)} \Gamma(\alpha+1) \left(1 + \frac{x}{\beta}\right)^{-(\alpha+1)} \\ &= \frac{\alpha}{\beta} \left(1 + \frac{x}{\beta}\right)^{-(\alpha+1)}, \end{aligned}$$

and we can conclude that $X \sim Lomax(\beta, \alpha)$.

Using this mixture representation, it is not difficult to see that the unconditional mean and variance of X are given by

$$E(X) = \beta E[\lambda^{-1}] = \frac{\beta}{\alpha - 1}, \quad \alpha > 1,$$

$$Var(X) = \beta^2 \left\{ E[\lambda^{-1}]^2 + Var[\lambda^{-1}] \right\} = \frac{\alpha \beta^2}{(\alpha - 1)^2(\alpha - 2)}, \quad \alpha > 2,$$

since $\lambda^{-1} \sim IG(\alpha, 1)$, where $IG(a, b)$ denotes the Inverse Gamma distribution with parameters a and b , mean $b/(a - 1)$, $a > 1$, and variance $b^2/(a - 1)^2(a - 2)$, $a > 2$.

Now, suppose that $\mathbf{X} = (X_1, \dots, X_n)$ is a random sample of size n from the Lomax distribution. We assume that the mixing parameters $\boldsymbol{\lambda} = (\lambda_1, \dots, \lambda_n)$ are a priori independent. The complete conditional distribution of $\boldsymbol{\lambda}$ using the hierarchical form is given by

$$\begin{aligned} f(\boldsymbol{\lambda}|\mathbf{x}, \beta, \alpha) &\propto f(\mathbf{x}|\beta, \boldsymbol{\lambda}) f(\boldsymbol{\lambda}|\alpha) \\ &\propto \prod_{i=1}^n \lambda_i \exp(-\lambda_i x_i / \beta) \prod_{i=1}^n \lambda_i^{\alpha-1} \exp(-\lambda_i) \\ &\propto \prod_{i=1}^n \lambda_i^{\alpha} \exp \left\{ -\lambda_i \left(1 + \frac{x_i}{\beta} \right) \right\}, \end{aligned}$$

so that

$$\lambda_i | \mathbf{x}, \boldsymbol{\lambda}_{-i}, \alpha, \beta \sim Gamma \left(\alpha + 1, 1 + \frac{x_i}{\beta} \right).$$

Again, using the hierarchical form, we obtain the complete conditional distributions of α and β as

$$\begin{aligned} f(\alpha | \mathbf{x}, \boldsymbol{\lambda}, \beta) &\propto f(\boldsymbol{\lambda}|\alpha) \pi(\beta, \alpha) \propto [\Gamma(\alpha)]^{-n} \left(\prod_{i=1}^n \lambda_i \right)^{\alpha-1} \pi(\beta, \alpha), \\ f(\beta | \mathbf{x}, \boldsymbol{\lambda}, \alpha) &\propto f(\mathbf{x}|\beta, \boldsymbol{\lambda}) \pi(\beta, \alpha) \propto \beta^{-n} \exp \left\{ -\frac{1}{\beta} \sum_{i=1}^n \lambda_i x_i \right\} \pi(\beta, \alpha). \end{aligned}$$

We note that, using this representation of the Lomax distribution, each observation X_i is associated with one mixing parameter λ_i , whose posterior mean or median can be used to identify a possible outlier.

3 Prior specification

We now complete the Bayesian model by specifying a prior distribution for α and β . We consider non-informative priors on these parameters and verify the existence of their posterior distribution.

3.1 Jeffreys prior

A commonly used objective prior in Bayesian analysis is Jeffreys prior [16], which is defined as

$$\pi_J(\beta, \alpha) \propto |I(\beta, \alpha)|^{1/2},$$

where $I(\cdot)$ stands for the Fisher information matrix. This is given by

$$I(\beta, \alpha) = n \begin{bmatrix} \frac{\alpha}{\beta^2(\alpha+2)} & -\frac{1}{\beta(\alpha+1)} \\ -\frac{1}{\beta(\alpha+1)} & \frac{1}{\alpha^2} \end{bmatrix},$$

from which we obtain

$$\pi_J(\beta, \alpha) \propto \frac{1}{\beta(\alpha+1)\alpha^{1/2}(\alpha+2)^{1/2}}, \quad \beta, \alpha > 0.$$

Considering independence between the parameters, the Jeffreys joint prior for (β, α) is given by $\pi_{IJ}(\beta, \alpha) \propto \pi(\beta)\pi(\alpha) = 1/\beta\alpha$.

Then, substituting $\pi(\beta, \alpha)$ in the expressions for the complete conditional densities, we obtain

$$f(\alpha|\mathbf{x}, \boldsymbol{\lambda}, \beta) \propto \frac{1}{(\alpha+1)\alpha^{1/2}(\alpha+2)^{1/2}\Gamma^n(\alpha)} \left(\prod_{i=1}^n \lambda_i \right)^{\alpha-1}$$

for the dependent Jeffreys prior and

$$f(\alpha|\mathbf{x}, \boldsymbol{\lambda}, \beta) \propto \frac{1}{\alpha\Gamma^n(\alpha)} \left(\prod_{i=1}^n \lambda_i \right)^{\alpha-1}$$

for the independence case. So, the complete conditional distribution of α is not of standard form and a Metropolis-Hastings algorithm [13, 10] is used to sample its values. The complete conditional density of β is given by

$$f(\beta|\mathbf{x}, \boldsymbol{\lambda}, \alpha) \propto \beta^{-(n+1)} \exp \left\{ -\frac{1}{\beta} \sum_{i=1}^n \lambda_i x_i \right\}$$

for both dependent and independent Jeffreys priors. It then follows that

$$\beta|\mathbf{x}, \boldsymbol{\lambda}, \alpha \sim IG \left(n, \sum_{i=1}^n \lambda_i x_i \right).$$

3.2 Reference prior

Reference priors were first proposed by [7] and further developed by [2, 3, 4, 11, 6, 5], among others. The idea is to specify a prior distribution such that,

even for moderate sample sizes, the information provided by the data should dominate the prior information. In particular, [3] discuss the construction of a non-informative prior that gives a different treatment for parameters of interest and nuisance parameters. When there are nuisance parameters (which is the case in this paper), one must establish an order parametrization between the interest and nuisance parameters. The following proposition is borrowed from [8] and adapted to the Lomax case.

Proposition 3.1. *Let $f(\mathbf{x}|\beta, \alpha)$, $(\beta, \alpha) \in \Delta \times \alpha(\beta) \subseteq \mathbb{R} \times \mathbb{R}$ be a probability model. Suppose that the joint posterior distribution of (β, α) is asymptotically normal with covariance matrix $S(\hat{\beta}, \hat{\alpha}) = I^{-1}(\hat{\beta}, \hat{\alpha})$, where $\hat{\beta}$ and $\hat{\alpha}$ are consistent estimators of β and α , respectively. Then, if β is the parameter of interest and α is the nuisance parameter,*

(i) *The conditional reference prior of α given β is*

$$\pi(\alpha|\beta) \propto [I_{22}(\beta, \alpha)]^{1/2}, \quad \alpha \in \alpha(\beta).$$

(ii) *If $\pi(\alpha|\beta)$ is not proper, a compact approximation $\{\alpha_i(\beta), i = 1, 2, \dots\}$ to $\alpha(\beta)$ is required and the reference prior of α given β is*

$$\pi_i(\alpha|\beta) = \frac{[I_{22}(\beta, \alpha)]^{1/2}}{\int_{\alpha_i(\beta)} [I_{22}(\beta, \alpha)]^{1/2} d\alpha}, \quad \alpha \in \alpha_i(\beta).$$

(iii) *The sequence of priors can be obtained as*

$$\pi_i(\beta) \propto \exp \left\{ \int_{\alpha_i(\beta)} \pi_i(\alpha|\beta) \log [s_{11}^{1/2}(\beta, \alpha)] d\alpha \right\},$$

$$\text{where } s_{11}^{1/2}(\beta, \alpha) = I_{\beta}(\beta, \alpha) = I_{11} - I_{12}I_{22}^{-1}I_{21}.$$

(iv) *The reference posterior distribution of β given data $\mathbf{x} = (x_1, \dots, x_n)$ is*

$$\pi(\beta|\mathbf{x}) \propto \pi(\beta) \left\{ \int_{\alpha(\beta)} \left[\prod_{i=1}^n f(x_i|\beta, \alpha) \right] \pi(\alpha|\beta) d\alpha \right\}.$$

Proof. See a heuristic justification in [8]. □

Corollary 3.1. *If the nuisance parameter space $\alpha(\beta) = \alpha$ is independent of β , and the functions $s_{11}^{-1/2}(\beta, \alpha)$ and $I_{22}^{1/2}(\beta, \alpha)$ factorize in the form*

$$[s_{11}(\beta, \alpha)]^{-1/2} = f_1(\beta) g_1(\alpha), \quad [I_{22}(\beta, \alpha)]^{1/2} = f_2(\beta) g_2(\alpha),$$

then

$$\pi(\beta) \propto f_1(\beta) \quad \text{and} \quad \pi(\alpha|\beta) \propto g_2(\alpha).$$

Thus, the reference prior relative to the ordered parametrization (β, α) is given by

$$\pi_{\mathbf{R}}(\beta, \alpha) = \pi(\beta)\pi(\alpha|\beta) = f_1(\beta)g_2(\alpha).$$

Proof. See proof of Theorem 12 in [8]. \square

Proposition 3.2. *The joint reference prior for the Lomax model with parameters β and α , $(\beta, \alpha) \in \Delta \times \alpha(\beta) \subseteq \mathbb{R} \times \mathbb{R}$, is given by*

$$\pi_{\mathbf{R}}(\beta, \alpha) \propto \frac{\pi(\beta)}{\alpha},$$

where $\pi(\beta) \propto f_1(\beta)$.

Proof. The inverse of the Fisher information matrix is given by

$$I^{-1}(\beta, \alpha) = S(\beta, \alpha) = \frac{1}{n} \begin{bmatrix} \frac{\beta^2(\alpha+2)(\alpha+1)^2}{\alpha} & \beta\alpha(\alpha+2)(\alpha+1) \\ \beta\alpha(\alpha+2)(\alpha+1) & \alpha^2(\alpha+1)^2 \end{bmatrix},$$

from which we obtain

$$[s_{11}(\beta, \alpha)]^{-1/2} \propto \left[\frac{\beta^2(\alpha+2)(\alpha+1)^2}{\alpha} \right]^{-1/2} = \frac{\alpha^{1/2}}{\beta(\alpha+2)^{1/2}(\alpha+1)}.$$

Considering Corollary 3.1,

$$f_1(\beta) = \frac{1}{\beta} \quad \text{and} \quad g_1(\alpha) = \frac{\alpha^{1/2}}{(\alpha+2)^{1/2}(\alpha+1)}.$$

From the Fisher information matrix, we also have that

$$[I_{22}(\beta, \alpha)]^{1/2} = \left[\frac{1}{\alpha^2} \right]^{1/2} = \frac{1}{\alpha},$$

and finally, $f_2(\beta) = 1$ and $g_2(\alpha) = 1/\alpha$. \square

So, considering β as the parameter of interest and α as the nuisance parameter, we conclude that the joint reference prior for (β, α) is given by

$$\pi_{\mathbf{R}}(\beta, \alpha) \propto f_1(\beta)g_2(\alpha) = \frac{1}{\alpha\beta}, \quad \beta, \alpha > 0.$$

Note that this reference prior coincides with the Jeffreys prior under the assumption of independence of parameters $(\pi_{\mathbf{IJ}}(\beta, \alpha))$ in the Lomax distribution. Consequently, the complete conditional distributions are the same as for the independent Jeffreys prior.

Applying the Bayes theorem, the joint posterior density is given by

$$\begin{aligned}\pi(\beta, \alpha | \mathbf{x}) &\propto \left(\frac{\alpha}{\beta}\right)^n \prod_{i=1}^n \left(1 + \frac{x_i}{\beta}\right)^{-(\alpha+1)} \times \frac{1}{\alpha\beta} \\ &\propto \alpha^{n-1} \beta^{-(n+1)} \prod_{i=1}^n \left(1 + \frac{x_i}{\beta}\right)^{-(\alpha+1)}.\end{aligned}\quad (1)$$

Proposition 3.3. *The posterior distribution (1) is improper for $n = 1$ and proper for $n > 1$.*

Proof. See Appendix A. □

4 Simulation study

In this section, we perform a Monte Carlo study to evaluate the methodology described in the previous section. We generated $m = 500$ replications of samples of sizes $n = 50, 100, 150, 200, 300$ and 500 from the Lomax distribution, considering parameter values $\beta = 2$ and $\alpha = 1.5$. The model was then estimated using Jeffreys and reference priors. We used the Metropolis-Hastings algorithm and the Gibbs sampler implemented in software R [21] to simulate two chains of values from the posterior distribution. A total of 11,000 iterations with jumps of 10 and a burn-in of 1,000 were performed for each chain, thus leading to a final sample of 1,000 values for each chain. Also, the Gelman and Rubin's Monte Carlo Markov Chain (MCMC) convergence diagnostic [12] provided in the R package CODA [20] was used to monitor convergence of the two parallel chains.

Let $\hat{\theta}^{(j)}$ be the estimate of parameter θ for the j -th replication, $j = 1, \dots, m$. These are the parameter posterior means calculated from the 2,000 simulated values for each replication. To evaluate the estimation method, two criteria were considered: the bias and square root of the mean square error (or simply root mean square error, rmse), which are defined as

$$\begin{aligned}\text{bias} &= \frac{1}{m} \sum_{j=1}^m \hat{\theta}^{(j)} - \theta, \\ \text{rmse} &= \sqrt{\frac{1}{m} \sum_{j=1}^m \left(\hat{\theta}^{(j)} - \theta\right)^2}.\end{aligned}$$

The results from the simulated experiment appear in Tables 1-4 and in Figure 1. Tables 1 and 3 show the computed posterior mean, standard deviation (SD), 95% credibility interval (CI), bias and rmse for each parameter and sample size, considering Jeffreys and reference priors, respectively. We note that the performances are barely similar whatever the prior we adopt. For the parameter

of interest β the bias is relatively small and negative for both priors, while for the nuisance parameter α the bias becomes positive for sample sizes $n \geq 100$ when a Jeffreys prior is adopted. With respect to accuracy, we obtained good results for β and α with relatively small rmse for both Jeffreys and reference priors and moderate sample sizes. Overall, the biases tend to reduce when moving from a Jeffreys to a reference prior, however this is mainly observed in the rmse for sample sizes $n \geq 200$ (see Figure 1). The Gelman and Rubin's diagnostic values presented in Tables 2 and 4 are close to 1, which means that the algorithm converged, independent of the initial values adopted. Note also from these tables that the acceptance rates were better (higher) when using the reference prior.

Table 1: Results of the simulation study using Jeffreys prior, considering true values of parameters: $\beta = 2$ and $\alpha = 1.5$.

sample size	parameter	mean	SD	95% CI	bias	rmse
50	β	3.5330	2.2255	[0.8639 ; 9.2080]	-1.5330	5.6185
	α	2.1413	0.9949	[0.8596 ; 4.6271]	-0.6413	2.5356
100	β	2.2017	0.9074	[1.0549 ; 4.2964]	-0.7017	1.1643
	α	1.7865	0.4875	[1.0966 ; 2.9148]	0.2135	0.5394
150	β	2.1693	0.6548	[1.1611 ; 3.7067]	-0.6693	0.9377
	α	1.6383	0.3316	[1.0897 ; 2.4156]	0.3616	0.4912
200	β	2.2161	0.6528	[1.2327 ; 3.7689]	-0.7161	1.2285
	α	1.9017	0.4812	[1.1631 ; 3.0344]	0.0983	0.8494
300	β	2.1263	0.4984	[1.3307 ; 3.2698]	-0.6263	0.9486
	α	1.8401	0.3697	[1.2423 ; 2.6795]	0.1599	0.6497
500	β	2.0498	0.3326	[1.4456 ; 2.7474]	-0.5498	0.6428
	α	1.4213	0.1508	[1.1496 ; 1.7245]	0.5787	0.5980

Table 2: Evaluation of the algorithm using Jeffreys prior.

sample size	50	100	150	200	300	500
acceptance rate	0.4553	0.3375	0.2799	0.2337	0.1311	0.1312
Gelman-Rubin	1.0033	1.0068	1.0045	1.0025	1.0003	1.0005

5 Application

In order to illustrate the methodology proposed in this paper, we consider a sample of computer file sizes (in bytes) for all 269 files with the *.ini extension on a Windows-based personal computer. These data can be downloaded from the website <http://web.uvic.ca/~dfiles/downloads/data>. A previous work by [14] has demonstrated the superiority of the Lomax distribution over several other competitors for modeling such file sizes. Those authors also pro-

Table 3: Results of the simulation study using reference prior, considering true values of parameters: $\beta = 2$ and $\alpha = 1.5$.

sample size	parameter	mean	SD	95% CI	bias	rmse
50	β	3.4805	1.9893	[0.8876 ; 8.2304]	-1.4805	4.2210
	α	2.0957	0.8955	[0.8705 ; 4.2278]	-0.5957	1.7641
100	β	2.7262	1.1961	[1.0864 ; 5.6940]	-0.7262	2.5608
	α	1.8305	0.5609	[1.0096 ; 3.1789]	-0.3306	1.2756
150	β	2.3929	0.8328	[1.1860 ; 4.4392]	-0.3929	1.4407
	α	1.6956	0.4042	[1.0791 ; 2.6592]	-0.1956	0.7083
200	β	2.2596	0.6629	[1.2539 ; 3.8479]	-0.2596	1.0196
	α	1.6162	0.3159	[1.1099 ; 2.3456]	-0.1161	0.4858
300	β	2.2290	0.5285	[1.3876 ; 3.4468]	-0.2290	0.8646
	α	1.6035	0.2534	[1.1825 ; 2.1718]	-0.1035	0.4137
500	β	2.0821	0.3685	[1.4618 ; 2.8919]	-0.0821	0.5267
	α	1.5378	0.1795	[1.2259 ; 1.9224]	-0.0378	0.2593

Table 4: Evaluation of algorithm using reference prior.

sample size	50	100	150	200	300	500
acceptance rate	0.9317	0.9109	0.8938	0.8785	0.8578	0.8151
Gelman-Rubin	1.0021	1.0003	1.0002	1.0477	1.0008	1.0019

vide technical information suggesting that the distribution should have infinite variance (i.e. $\alpha < 2$) in this context.

The samples for the Jeffreys and reference posterior distributions of the parameters β and α were obtained by the Gibbs sampler and Metropolis-Hastings algorithm, i.e. through MCMC methods implemented in software R; see Appendix B. The convergence of the chains were tested by using the Gelman and Rubin method implemented in the R package CODA. Graphical traces of those methods and kernel density estimation for each parameter showed that there were no convergence problems. We generated two parallel chains of size 80,000 for each parameter. The first 20,000 iterations were ignored to eliminate the effect of the initial values (burn-in) and, to avoid correlation problems, we considered a spacing of size 20, obtaining a sample of size 3,000.

The posterior results from using both the Jeffreys and reference priors are shown in Table 5. It may be noticed that the posterior results are all very similar.

In Figures 2 and 3 we show plots of the generated samples and the empirical marginal posteriors for model parameters β and α , based on the generated chains of the marginal Jeffreys and reference posteriors, respectively.

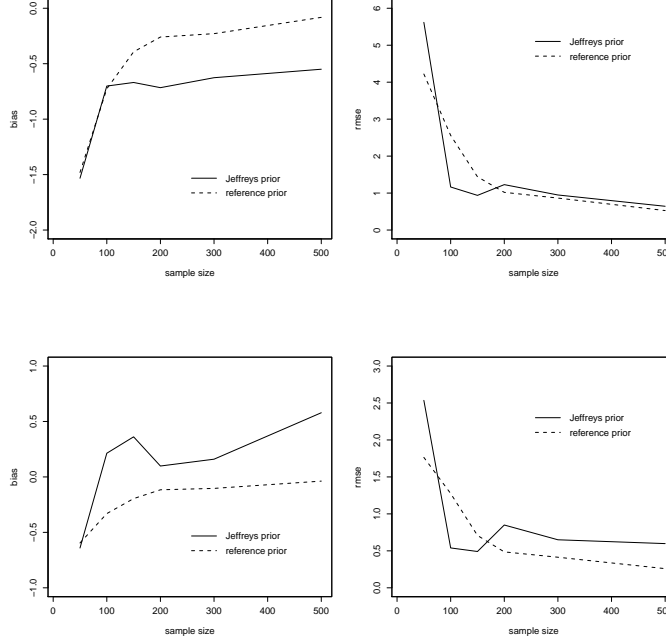


Figure 1: Bias and rmse for β (upper panels) and α (lower panels) parameters using Jeffreys and reference priors.

6 Concluding remarks

In this paper we evaluated the Bayesian method to estimate the parameters in a Lomax distribution under two non-informative prior specifications. We showed that the joint posterior distribution is proper no matter which non-informative prior is used. We also obtained a scale mixture representation of the Lomax distribution in which the complete conditional distribution of the scale parameter is of known closed form and easy to sample. As a by product, this representation allows for the mixing parameters to be used to identify possible outliers. Overall, the results obtained indicate that the Bayesian method estimates the

Table 5: Posterior summaries for the Lomax parameters using Jeffreys and reference priors.

prior	parameter	mean	SD	95% CI
π_J	β	131.1242	24.5318	[88.9900 ; 184.6600]
	α	0.5008	0.0435	[0.4207 ; 0.5920]
π_R	β	130.4562	23.9599	[90.0000 ; 182.7000]
	α	0.4986	0.0424	[0.4226 ; 0.5865]

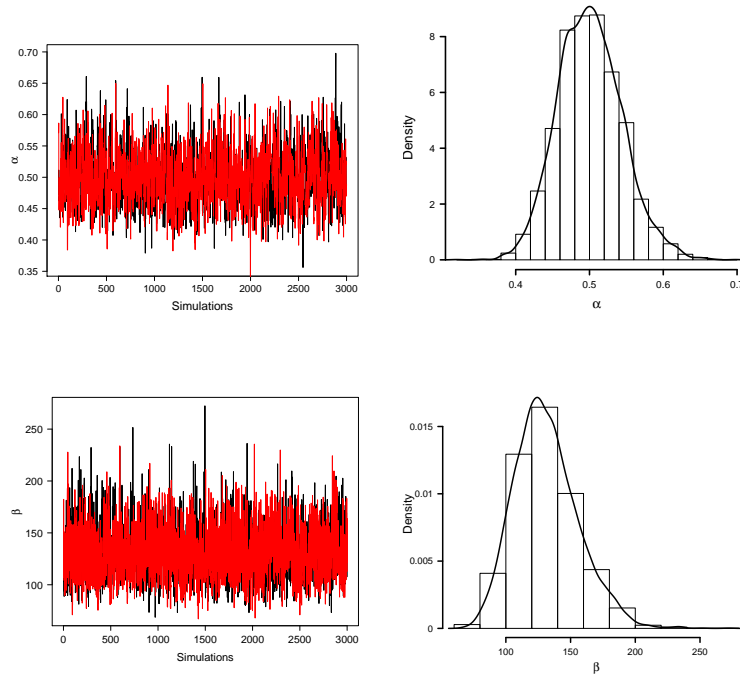


Figure 2: Trace and density of α (upper panels) and β (lower panels) using Jeffreys prior.

parameters well under both prior specifications if the sample size is not too small. Of course, as in any Monte Carlo study, our results are limited to our particular selection of sample sizes and prior distributions. We hope that our findings are useful to the practitioners.

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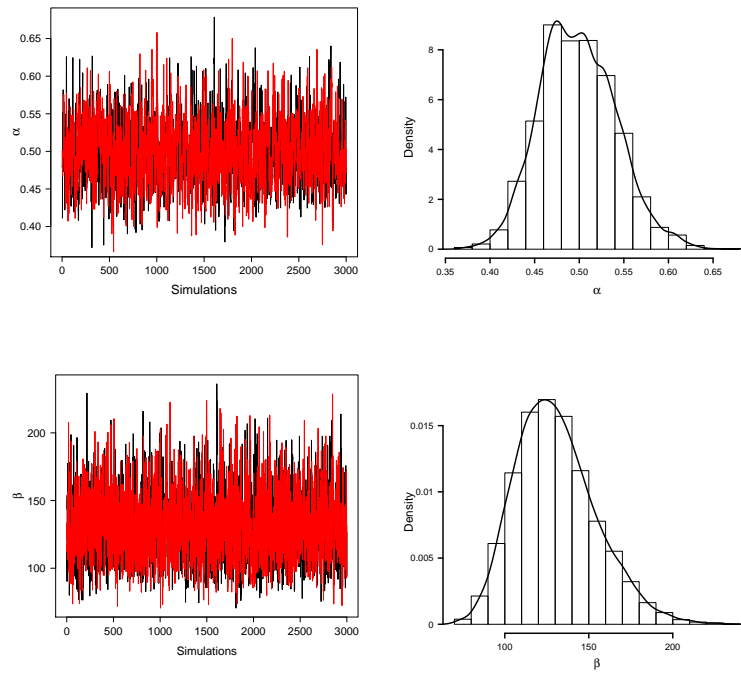


Figure 3: Trace and density of α (upper panels) and β (lower panels) using reference prior.

A Appendix A

A.1 Verifying that the posterior is proper under Jeffreys prior

Under Jeffreys prior, the joint posterior density of β and α is given by

$$\pi(\beta, \alpha | \mathbf{x}) \propto \frac{\alpha^{n-1/2} \beta^{-(n+1)}}{(\alpha+1)(\alpha+2)^{1/2}} \prod_{i=1}^n \left(1 + \frac{x_i}{\beta}\right)^{-(\alpha+1)}.$$

We next show that the integral of this expression is finite for any sample size n .

(a) Verifying for $n = 1$:

$$\begin{aligned} & \int_0^\infty \int_0^\infty \frac{\alpha^{1/2} \beta^{-2}}{(\alpha+1)(\alpha+2)^{1/2}} \left(1 + \frac{x}{\beta}\right)^{-(\alpha+1)} d\beta d\alpha \\ &= \int_0^\infty \frac{\alpha^{1/2}}{(\alpha+1)(\alpha+2)^{1/2}} \left(\int_0^\infty \beta^{-2} \left(1 + \frac{x}{\beta}\right)^{-(\alpha+1)} d\beta \right) d\alpha \\ &= \int_0^\infty \frac{\alpha^{1/2}}{(\alpha+1)(\alpha+2)^{1/2}} \left(\int_0^\infty \beta^{-2} \left(\frac{\beta+x}{\beta}\right)^{-(\alpha+1)} d\beta \right) d\alpha \\ &= \int_0^\infty \frac{\alpha^{1/2}}{(\alpha+1)(\alpha+2)^{1/2}} \left(\int_0^\infty \beta^{-2} \beta^{\alpha+1} (\beta+x)^{-(\alpha+1)} d\beta \right) d\alpha \\ &= \int_0^\infty \frac{\alpha^{1/2}}{(\alpha+1)(\alpha+2)^{1/2}} \left(\int_0^\infty \beta^{\alpha-1} (\beta+x)^{-(\alpha+1)} d\beta \right) d\alpha \\ &= \int_0^\infty \frac{\alpha^{1/2}}{(\alpha+1)(\alpha+2)^{1/2}} \frac{1}{x\alpha} d\alpha \\ &= \frac{1}{x} \int_0^\infty \frac{\alpha^{-1/2}}{(\alpha+1)(\alpha+2)^{1/2}} d\alpha = \frac{\pi}{2x} < \infty. \end{aligned}$$

(b) Verifying for $n > 1$: First, we solve

$$\int_0^\infty \beta^{-(n+1)} \prod_{i=2}^n \left(1 + \frac{x_i}{\beta}\right)^{-(\alpha+1)} d\beta. \quad (2)$$

Consider $y = \min(x_2, \dots, x_n)$. Then, it follows that

$$\begin{aligned} \left(1 + \frac{x_i}{\beta}\right)^{\alpha+1} &\geq \left(1 + \frac{y}{\beta}\right)^{\alpha+1}, \quad \alpha > 0, \quad i = 2, \dots, n \\ \prod_{i=2}^n \left(1 + \frac{x_i}{\beta}\right)^{\alpha+1} &\geq \left(1 + \frac{y}{\beta}\right)^{(n-1)(\alpha+1)} \\ \prod_{i=2}^n \left(1 + \frac{x_i}{\beta}\right)^{-(\alpha+1)} &< \left(1 + \frac{y}{\beta}\right)^{-(n-1)(\alpha+1)}. \end{aligned}$$

Therefore,

$$\begin{aligned}
& \int_0^\infty \beta^{-(n+1)} \prod_{i=2}^n \left(1 + \frac{x_i}{\beta}\right)^{-(\alpha+1)} d\beta < \int_0^\infty \beta^{-(n+1)} \left(1 + \frac{y}{\beta}\right)^{-(n-1)(\alpha+1)} d\beta \\
&= \int_0^\infty \beta^{-(n+1)} \left(\frac{\beta+y}{\beta}\right)^{-(n-1)(\alpha+1)} d\beta \\
&= \int_0^\infty \beta^{-(n+1)} \beta^{n\alpha+n-\alpha-1} (\beta+y)^{-(n-1)(\alpha+1)} d\beta \\
&= \int_0^\infty \beta^{n\alpha-\alpha-2} (\beta+y)^{-(n-1)(\alpha+1)} d\beta = \frac{(n-1)!\Gamma(n\alpha-\alpha-1)}{y^n \Gamma(n\alpha-\alpha+n-1)}.
\end{aligned}$$

Then,

$$\begin{aligned}
& \int_0^\infty \int_0^\infty \frac{\alpha^{n-1/2} \beta^{-(n+1)}}{(\alpha+1)(\alpha+2)^{1/2}} \prod_{i=2}^n \left(1 + \frac{x_i}{\beta}\right)^{-(\alpha+1)} d\beta d\alpha \\
&= \int_0^\infty \frac{\alpha^{n-1/2}}{(\alpha+1)(\alpha+2)^{1/2}} \frac{(n-1)!\Gamma(n\alpha-\alpha-1)}{y^n \Gamma(n\alpha-\alpha+n-1)} d\alpha \\
&= \frac{(n-1)!}{y^n} \int_0^\infty \frac{\alpha^{n-1/2}}{(\alpha+1)(\alpha+2)^{1/2}} \frac{\Gamma(n\alpha-\alpha-1)}{\Gamma(n\alpha-\alpha+n-1)} d\alpha \\
&= \frac{(n-1)!}{y^n} \int_0^\infty \frac{\alpha^{n-1/2}}{(\alpha+1)(\alpha+2)^{1/2}} \frac{1}{\prod_{j=-1}^{n-2} (n\alpha-\alpha+j)} d\alpha. \quad (3)
\end{aligned}$$

But note that $(n\alpha-\alpha+j) \geq (n\alpha-\alpha)$, $\alpha > 0$, $j \geq -1$. Therefore,

$$(n\alpha-\alpha+j)^{-1} < (n\alpha-\alpha)^{-1}$$

$$\prod_{j=-1}^{n-2} (n\alpha-\alpha+j)^{-1} < \prod_{j=-1}^{n-2} (n\alpha-\alpha)^{-1}$$

$$\prod_{j=-1}^{n-2} (n\alpha-\alpha+j)^{-1} < (n\alpha-\alpha)^{-n+2}$$

and replacing in (3), we have that

$$\begin{aligned}
& \frac{(n-1)!}{y^n} \int_0^\infty \frac{\alpha^{n-1/2}}{(\alpha+1)(\alpha+2)^{1/2}} \frac{1}{\prod_{j=-1}^{n-2} (n\alpha-\alpha+j)} d\alpha \\
&< \frac{(n-1)!}{y^n} \int_0^\infty \frac{\alpha^{n-1/2}}{(\alpha+1)(\alpha+2)^{1/2}} (n\alpha-\alpha)^{-n+2} d\alpha = \infty.
\end{aligned}$$

We can conclude that the posterior distribution using Jeffreys prior is proper for $n \geq 1$.

A.2 Verifying that the posterior is proper under reference prior

Using a reference prior, the joint posterior density of β and α is given by

$$\pi(\beta, \alpha | \mathbf{x}) \propto \alpha^{n-1} \beta^{-(n+1)} \prod_{i=1}^n \left(1 + \frac{x_i}{\beta}\right)^{-(\alpha+1)}.$$

We next verify whether the integral of this expression is finite.

a) Verifying for $n = 1$:

$$\begin{aligned} \int_0^\infty \int_0^\infty \beta^{-2} \left(1 + \frac{x}{\beta}\right)^{-(\alpha+1)} d\beta d\alpha &= \int_0^\infty \int_0^\infty \beta^{-2} \left(\frac{\beta+x}{\beta}\right)^{-(\alpha+1)} d\beta d\alpha \\ &= \int_0^\infty \int_0^\infty \beta^{-2} \beta^{\alpha+1} (\beta+x)^{-(\alpha+1)} d\beta d\alpha \\ &= \int_0^\infty \int_0^\infty \beta^{\alpha-1} (\beta+x)^{-(\alpha+1)} d\beta d\alpha \\ &= \int_0^\infty \frac{1}{x\alpha} d\alpha = \infty. \end{aligned}$$

Therefore, the posterior distribution using a reference prior is improper for $n = 1$.

b) Verifying for $n > 1$: Note that

$$\int_0^\infty \beta^{-(n+1)} \prod_{i=2}^n \left(1 + \frac{x_i}{\beta}\right)^{-(\alpha+1)} d\beta$$

is the same integral (2) previously resolved. Then,

$$\begin{aligned} &\int_0^\infty \int_0^\infty \alpha^{n-1} \beta^{-(n+1)} \prod_{i=2}^n \left(1 + \frac{x_i}{\beta}\right)^{-(\alpha+1)} d\beta d\alpha \\ &= \frac{(n-1)!}{y^n} \int_0^\infty \alpha^{n-1} \frac{1}{\prod_{j=-1}^{n-2} (n\alpha - \alpha + j)} d\alpha \\ &< \frac{(n-1)!}{y^n} \int_0^\infty \alpha^{n-1} (n\alpha - \alpha)^{-n+2} d\alpha = \infty. \end{aligned}$$

Thus, the posterior distribution using reference prior is proper for $n > 1$.

B R code of the Metropolis-Hasting within Gibbs

```
#####  
#### clean memory  
rm(list=ls(all=T))  
graphics.off()  
#####  
dat=read.table("dataset1.txt",header=T,sep=",",quote="\",dec=".",  
              fill=T,na.strings="NA")  
n_dat=nrow(dat)  
#####  
#### Jeffreys (independence) and reference priors  
## $p(\alpha,\beta) \propto \frac{1}{\alpha \beta}$  
# use of the log of the conditional distribution of alpha  
alpha_cond=function(x,lambda,tuning){  
  repeat{  
    # xn: proposal value  
    xn=x+rnorm(1,0,tuning)  
    if(xn>0)  
      break  
  }  
  A1=length(lambda)*(lgamma(x)-lgamma(xn))+(xn-x)*  
    sum(log(lambda))+log(x)-log(xn)  
  A2=pnorm(x,log.p=T)-pnorm(xn,log.p=T)  
  A=A1+A2  
  if(log(runif(1))<=A) x=xn else x=x  
  return(x)  
}  
#####  
# some variable declaration  
burn=20000  
jump=20  
n.amostra=3000  
iter=burn+jump*n.amostra  
#  
ref_full=list(alpha1=numeric(),alpha2=numeric(),beta1=numeric(),  
              beta2=numeric())  
#  
#####  
# MCMC method: Gibbs sampler with Metropolis-Hastings  
# initial values  
alpha_iter=rgamma(1,1,1)  
beta_iter=rgamma(1,1,1)  
#  
tuning=3/3
```



```

#
for(j1 in 1:iter){
  # auxiliary variable
  lambda_aux=unlist(lapply(1:n_dat,function(i) rgamma(1,
    shape=(alpha_iter+1),rate=((dat[i,1]/beta_iter)+1))))
  # conditional distribution of beta
  beta_iter=1/rgamma(1,shape=n_dat,scale=1/sum(lambda_aux*dat))
  # generation of values of alpha from its conditional
  # distribution
  alphan=alpha_cond(alpha_iter,lambda_aux,tuning)
  #
  alpha_iter=alphan
  #
  ref_full$alpha1[j1]=alpha_iter
  ref_full$beta1[j1]=beta_iter
}
#=====

```

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